Variational Gaussian Processes

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Spotify

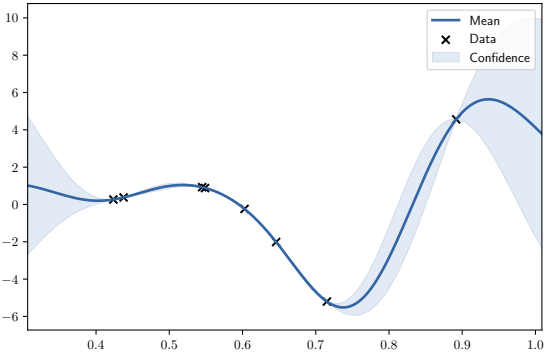
15 September 2020 @GPSS 2020

Gaussian process

Input and Output Data:

$$\mathbf{y} = (y_1, \ldots, y_N), \quad \mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)^\top$$

$$p(\mathbf{y}|\mathbf{f}) = \mathcal{N}\left(\mathbf{y}|\mathbf{f}, \sigma^2\mathbf{I}\right), \quad p(\mathbf{f}|\mathbf{X}) = \mathcal{N}\left(\mathbf{f}|0, \mathbf{K}(\mathbf{X}, \mathbf{X})\right)$$



Behind a Gaussian process fit

- Maximum likelihood estimate of the hyper-parameters.

$$\theta^* = \arg\max_\theta \log p(\mathbf{y}|\mathbf{X}, \theta) = \arg\max_\theta \log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K} + \sigma^2\mathbf{I}\right)$$

- Prediction on a test point given the observed data and the optimized hyper-parameters.

$$p(\mathbf{f}_*|\mathbf{X}_*, \mathbf{y}, \mathbf{X}, \theta) = \mathcal{N}\left(\mathbf{f}_*|\mathbf{K}_*(\mathbf{K} + \sigma^2\mathbf{I})^{-1}\mathbf{y}, \mathbf{K}_{**} - \mathbf{K}_*(\mathbf{K} + \sigma^2\mathbf{I})^{-1}\mathbf{K}_*^\top\right)$$

How to implement the log-likelihood (1)

- Compute the covariance matrix $\mathbf{K}$:

$$\mathbf{K} = \begin{pmatrix} k(\mathbf{x}_1, \mathbf{x}_1) & \cdots & k(\mathbf{x}_1, \mathbf{x}_N) \\ \vdots & \ddots & \vdots \\ k(\mathbf{x}_N, \mathbf{x}_1) & \cdots & k(\mathbf{x}_N, \mathbf{x}_N) \end{pmatrix}$$

  where $k(\mathbf{x}_i, \mathbf{x}_j) = \gamma \exp\left(-\frac{1}{2l^2}(\mathbf{x}_i - \mathbf{x}_j)^\top (\mathbf{x}_i - \mathbf{x}_j)\right)$

- The complexity is $O(N^2 Q)$.

How to implement the log-likelihood (2)

- Plug in the log-pdf of multi-variate normal distribution:

$$\begin{aligned}\log p(\mathbf{y}|\mathbf{X}) &= \log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K}+\sigma^2\mathbf{I}\right) \\ &= -\frac{1}{2}\log|2\pi(\mathbf{K}+\sigma^2\mathbf{I})| - \frac{1}{2}\mathbf{y}^\top(\mathbf{K}+\sigma^2\mathbf{I})^{-1}\mathbf{y} \\ &= -\frac{N}{2}N\log 2\pi - \sum_i \log \mathbf{L}_{ii} - \frac{1}{2}||\mathbf{L}^{-1}\mathbf{y}||^2\end{aligned}$$

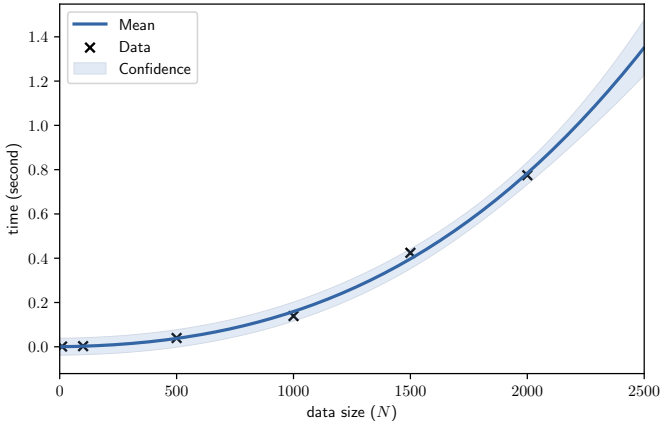
- Take a Cholesky decomposition: $\mathbf{L} = \texttt{chol}(\mathbf{K}+\sigma^2\mathbf{I})$, such that $\mathbf{K}+\sigma^2\mathbf{I} = \mathbf{L}\mathbf{L}^\top$.
- The computational complexity is $O(N^3+N^2+N)$. Therefore, the overall complexity including the computation of $\mathbf{K}$ is $O(N^3)$.

A quick profiling (N=1000, Q=10)

```
Line #    Time(ms)   % Time      Line Contents
 2                            def log_likelihood(kern, X, Y, sigma2):
 3           6.0       0.0        N = X.shape[0]
 4       55595.0      58.7        K = kern.K(X)
 5        4369.0       4.6        Ky = K + np.eye(N)*sigma2
 6       30012.0      31.7        L = np.linalg.cholesky(Ky)
 7        4361.0       4.6        LinvY = dtrtrs(L, Y, lower=1)[0]
 8          49.0       0.1        logL = N*np.log(2*np.pi)/-2.
 9          82.0       0.1        logL += np.square(LinvY).sum()/-2.
10         208.0       0.2        logL += -np.log(np.diag(L)).sum()
11           2.0       0.0        return logL
```

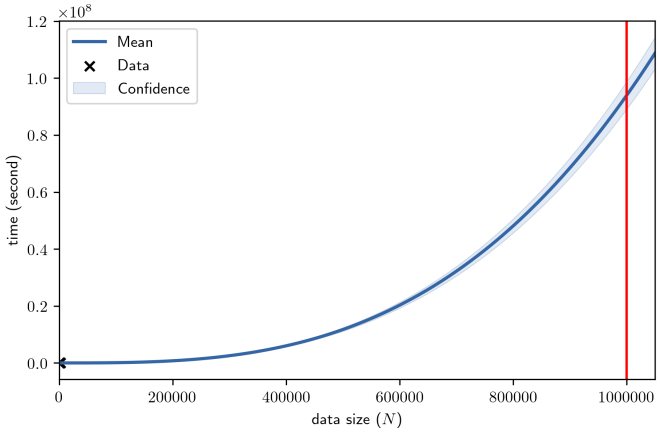
Empirical analysis of computational time

- I collect the run time for $N = \{10, 100, 500, 1000, 1500, 2000\}$.
- They take 1.3ms, 8.5ms, 28ms, 0.12s, 0.29s, 0.76s.



What if we have 1 million data points?

The mean of predicted computational time is 9.4×10^7 seconds ≈ 2.98 years.

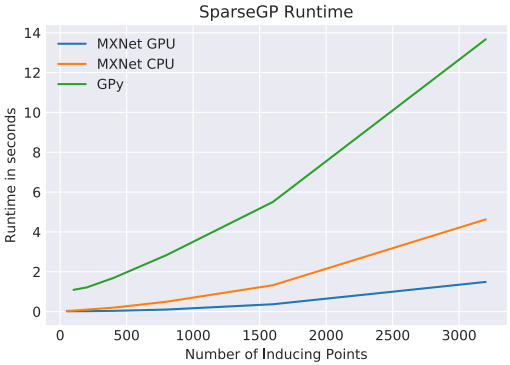


What about waiting for faster computers?

- Computational time = amount of work / computer speed.
- If the computer speed increase at the pace of 20% year over year:
  - After 10 years, it will take about 176 days.
  - After 50 years, it will take about 2.9 hours.

What about parallel computing / GPU?

- Ongoing works about speeding up Cholesky decomposition with multi-core CPU or GPU.
- Main limitation:
  - heavy communication and shared memory.
  - $O(N^2)$ memory consumption

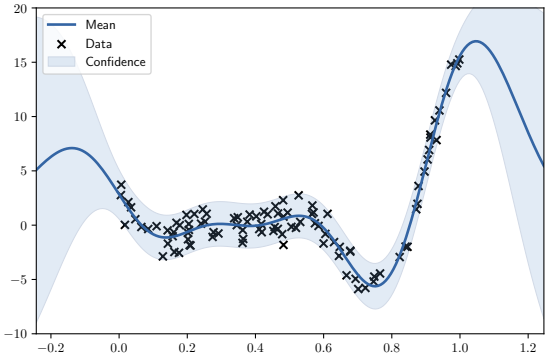


Other approaches

- Apart from speeding up the exact computation, there have been a lot of works on approximation of GP inference.
- These methods often target at some specific scenario and provide good approximation for the targeted scenarios.
- Provide an overview about common approximations.

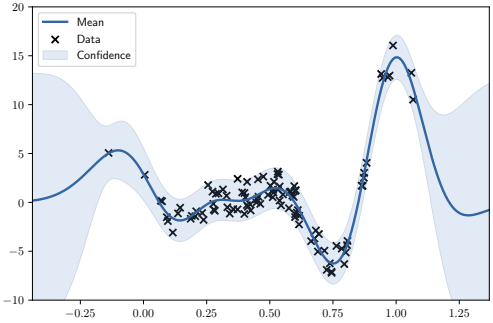
Big data (?)

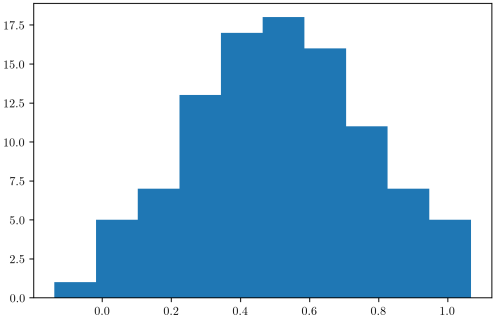
- lots of data $\neq$ complex function
- In real world problems, we often collect a lot of data for modeling relatively simple relations.



Data subsampling?

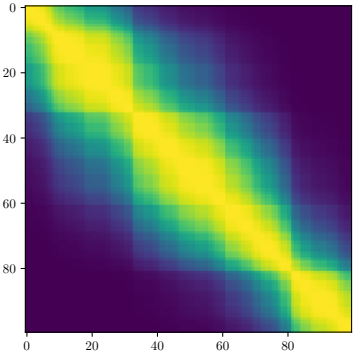
- Real data often do not evenly distributed.
- We tend to get a lot of data on common cases and very few data on rare cases.

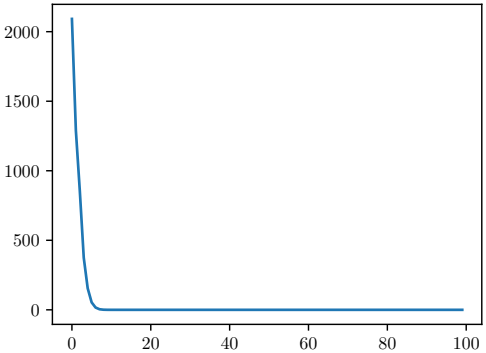




Covariance matrix of redundant data

- With redundant data, the covariance matrix becomes low rank.
- What about low rank approximation?





Low-rank approximation

- Let's recall the log-likelihood of GP:

$$\log p(\mathbf{y}|\mathbf{X}) = \log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K} + \sigma^2 \mathbf{I}\right),$$

  where $\mathbf{K}$ is the covariance matrix computed from $\mathbf{X}$ according to the kernel function $k(\cdot, \cdot)$ and σ^2 is the variance of the Gaussian noise distribution.
- Assume $\mathbf{K}$ to be low rank.
- This leads to Nyström approximation by Williams and Seeger [Williams and Seeger, 2001].

Approximation by subset

- Let's randomly pick a subset from the training data: $\mathbf{Z} \in \mathbb{R}^{M \times Q}$.
- Approximate the covariance matrix $\mathbf{K}$ by $\tilde{\mathbf{K}}$.

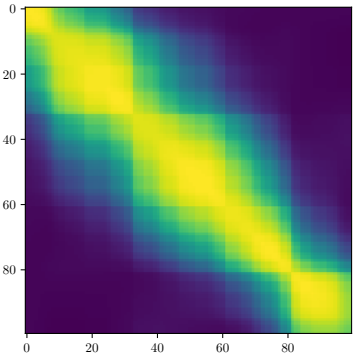
$$\tilde{\mathbf{K}} = \mathbf{K}_z \mathbf{K}_{zz}^{-1} \mathbf{K}_z^\top, \text{ where } \mathbf{K}_z = \mathbf{K}(\mathbf{X}, \mathbf{Z}) \text{ and } \mathbf{K}_{zz} = \mathbf{K}(\mathbf{Z}, \mathbf{Z}).$$

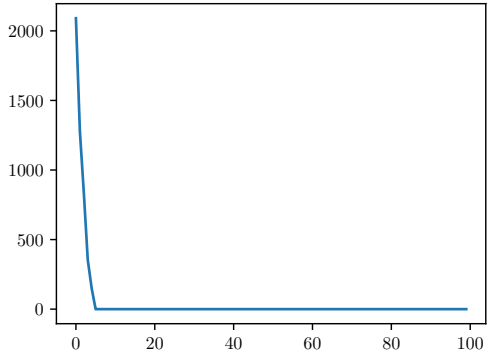
- Note that $\tilde{\mathbf{K}} \in \mathbb{R}^{N \times N}$, $\mathbf{K}_z \in \mathbb{R}^{N \times M}$ and $\mathbf{K}_{zz} \in \mathbb{R}^{M \times M}$.
- The log-likelihood is approximated by

$$\log p(\mathbf{y}|\mathbf{X}, \theta) \approx \log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K}_z \mathbf{K}_{zz}^{-1} \mathbf{K}_z^\top + \sigma^2 \mathbf{I}\right).$$

Nyström approximation example

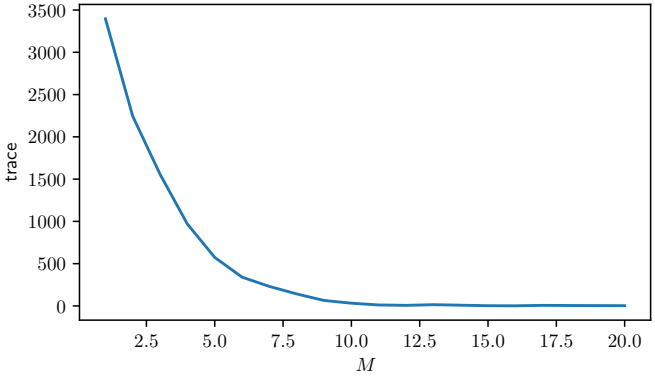
The covariance matrix with Nyström approximation using 5 random data points:





Nyström approximation example

Compute $\text{tr}\left(\mathbf{K} - \tilde{\mathbf{K}}\right)$ with different M.



Efficient computation using Woodbury formula

- The naive formulation does not bring any computational benefits.

$$\tilde{\mathcal{L}} = -\frac{1}{2}\log|2\pi(\tilde{\mathbf{K}} + \sigma^2\mathbf{I})| - \frac{1}{2}\mathbf{y}^\top(\tilde{\mathbf{K}} + \sigma^2\mathbf{I})^{-1}\mathbf{y}$$

- Apply the Woodbury formula:

$$(\mathbf{K}_z\mathbf{K}_{zz}^{-1}\mathbf{K}_z^\top + \sigma^2\mathbf{I})^{-1} = \sigma^{-2}\mathbf{I} - \sigma^{-4}\mathbf{K}_z(\mathbf{K}_{zz} + \sigma^{-2}\mathbf{K}_z^\top\mathbf{K}_z)^{-1}\mathbf{K}_z^\top$$

- Note that $(\mathbf{K}_{zz} + \sigma^{-2}\mathbf{K}_z^\top\mathbf{K}_z) \in \mathbb{R}^{M\times M}$.
- The computational complexity reduces to $O(NM^2)$.

Nyström approximation

- The approximation is directly done on the covariance matrix without the concept of pseudo data.
- The approximation becomes exact if the whole data set is taken, *i.e.*, $\mathbf{K}\mathbf{K}^{-1}\mathbf{K}^{\top} = \mathbf{K}$.
- The subset selection is done randomly.

Gaussian process with Pseudo Data (1)

- Snelson and Ghahramani [2006] proposes the idea of having pseudo data, which is later referred to as Fully independent training conditional (FITC).
- Augment the training data $(\mathbf{X}, \mathbf{y})$ with pseudo data $\mathbf{u}$ at location $\mathbf{Z}$.

$$p\left(\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} \mid \begin{bmatrix}\mathbf{X}\\ \mathbf{Z}\end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} | 0, \begin{bmatrix}\mathbf{K}_{ff} + \sigma^2\mathbf{I} & \mathbf{K}_{fu}\\ \mathbf{K}_{fu}^\top & \mathbf{K}_{uu}\end{bmatrix}\right)$$

where $\mathbf{K}_{ff} = \mathbf{K}(\mathbf{X}, \mathbf{X})$, $\mathbf{K}_{fu} = \mathbf{K}(\mathbf{X}, \mathbf{Z})$ and $\mathbf{K}_{uu} = \mathbf{K}(\mathbf{Z}, \mathbf{Z})$.

Gaussian process with Pseudo Data (2)

- Thanks to the marginalization property of Gaussian distribution,

$$p(\mathbf{y}|\mathbf{X}) = \int_{\mathbf{u}} p(\mathbf{y}, \mathbf{u}|\mathbf{X}, \mathbf{Z}).$$

- Further re-arrange the notation:

$$p(\mathbf{y}, \mathbf{u}|\mathbf{X}, \mathbf{Z}) = p(\mathbf{y}|\mathbf{u}, \mathbf{X}, \mathbf{Z})p(\mathbf{u}|\mathbf{Z})$$

where $p(\mathbf{u}|\mathbf{Z}) = \mathcal{N}(\mathbf{u}|0, \mathbf{K}_{uu})$,
$p(\mathbf{y}|\mathbf{u}, \mathbf{X}, \mathbf{Z}) = \mathcal{N}\left(\mathbf{y}|\mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{u}, \mathbf{K}_{ff} - \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^{\top} + \sigma^2\mathbf{I}\right)$.

FITC approximation (1)

- So far, $p(\mathbf{y}|\mathbf{X})$ has not been changed, but there is no speed-up.
- $\mathbf{K}_{ff} \in \mathbb{R}^{N \times N}$ in $\mathbf{K}_{ff} - \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^{\top} + \sigma^2\mathbf{I}$.
- The FITC approximation assumes

$$\tilde{p}(\mathbf{y}|\mathbf{u}, \mathbf{X}, \mathbf{Z}) = \mathcal{N}\left(\mathbf{y}|\mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{u}, \mathbf{\Lambda} + \sigma^2\mathbf{I}\right),$$

where $\mathbf{\Lambda} = (\mathbf{K}_{ff} - \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^{\top}) \circ \mathbf{I}$.

FITC approximation (2)

- Marginalize $\mathbf{u}$ from the model definition:

$$\tilde{p}(\mathbf{y}|\mathbf{X},\mathbf{Z}) = \mathcal{N}\left(\mathbf{y}|0, \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^{\top} + \mathbf{\Lambda} + \sigma^2\mathbf{I}\right)$$

- Woodbury formula can be applied in the sam way as in Nyström approximation:

$$(\mathbf{K}_z\mathbf{K}_{zz}^{-1}\mathbf{K}_z^{\top} + \mathbf{\Lambda} + \sigma^2\mathbf{I})^{-1} = \mathbf{A} - \mathbf{A}\mathbf{K}_z(\mathbf{K}_{zz} + \mathbf{K}_z^{\top}\mathbf{A}\mathbf{K}_z)^{-1}\mathbf{K}_z^{\top}\mathbf{A},$$

where $\mathbf{A} = (\mathbf{\Lambda} + \sigma^2\mathbf{I})^{-1}$.

FITC approximation (3)

- FITC allows the pseudo data not being a subset of training data.
- The inducing inputs $\mathbf{Z}$ can be optimized via gradient optimization.
- Like Nyström approximation, when taking all the training data as inducing inputs, the FITC approximation is equivalent to the original GP:

$$\tilde{p}(\mathbf{y}|\mathbf{X}, \mathbf{Z} = \mathbf{X}) = \mathcal{N}\left(\mathbf{y}|0, \mathbf{K}_{ff} + \sigma^2\mathbf{I}\right)$$

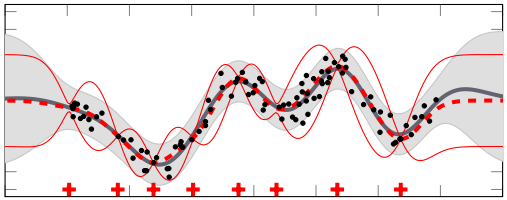
- FITC can be combined easily with expectation propagation (EP).
- Bui et al. [2017] provides an overview and a nice connection with variational sparse GP.

Model Approximation vs. Approximate Inference

FITC approximation changes the model definition.

- A better objective under FITC does not necessarily corresponds to a better approximation to the original GP.
- In fact, optimizing $\mathbf{Z}$ can lead to overfitting. [Quiñonero-Candela and Rasmussen, 2005, Bauer et al., 2016]

FITC (nlml $= 23.16$, $\sigma_n = 1.93 \cdot 10^{-4}$)



Optimal values for the exact GP: nlml $= 34.15$, $\sigma = 0.274$. [Bauer et al., 2016]

Model Approximation vs. Approximate Inference

Variational inference (VI) takes a different approach.

- VI keeps the model definition untouched.
- VI derives a lower bound of the log-marginal likelihood:

$$\log(y) \geq \int q(x) \log \frac{p(y,x)}{q(x)} \mathrm{d}x = \mathcal{L}$$

- Alternatively, it can be written as

$$\mathrm{KL}\left(q(x) \,\|\, p(x|y)\right) = \log p(y) - \mathcal{L}.$$

Variational Sparse Gaussian Process (1)

- Titsias [2009] introduces a variational approach for sparse GP.
- It follows the same concept of pseudo data:

$$p(\mathbf{y}|\mathbf{X}) = \int_{\mathbf{f},\mathbf{u}} p(\mathbf{y}|\mathbf{f})p(\mathbf{f}|\mathbf{u},\mathbf{X},\mathbf{Z})p(\mathbf{u}|\mathbf{Z})$$

where $p(\mathbf{u}|\mathbf{Z}) = \mathcal{N}(\mathbf{u}|0,\mathbf{K}_{uu})$,
$p(\mathbf{y}|\mathbf{u},\mathbf{X},\mathbf{Z}) = \mathcal{N}\left(\mathbf{y}|\mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{u}, \mathbf{K}_{ff} - \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^{\top} + \sigma^2\mathbf{I}\right)$.

Variational Sparse Gaussian Process (2)

- Instead of approximate the model, Titsias [2009] derives a variational lower bound.
- Normally, a variational lower bound of a marginal likelihood looks like

$$\begin{aligned}\log p(\mathbf{y}|\mathbf{X}) &= \log \int_{\mathbf{f},\mathbf{u}} p(\mathbf{y}|\mathbf{f}) p(\mathbf{f}|\mathbf{u},\mathbf{X},\mathbf{Z}) p(\mathbf{u}|\mathbf{Z}) \\ &\geq \int_{\mathbf{f},\mathbf{u}} q(\mathbf{f},\mathbf{u}) \log \frac{p(\mathbf{y}|\mathbf{f}) p(\mathbf{f}|\mathbf{u},\mathbf{X},\mathbf{Z}) p(\mathbf{u}|\mathbf{Z})}{q(\mathbf{f},\mathbf{u})}.\end{aligned}$$

Special Variational Posterior

- Titsias [2009] defines an unusual variational posterior:

$$q(\mathbf{f}, \mathbf{u}) = p(\mathbf{f}|\mathbf{u}, \mathbf{X}, \mathbf{Z})q(\mathbf{u}), \quad \text{where } q(\mathbf{u}) = \mathcal{N}(\mathbf{u}|\mu, \Sigma).$$

- Plug it into the lower bound:

$$\begin{aligned}
\mathcal{L} &= \int_{\mathbf{f},\mathbf{u}} p(\mathbf{f}|\mathbf{u}, \mathbf{X}, \mathbf{Z})q(\mathbf{u}) \log \frac{p(\mathbf{y}|\mathbf{f})\cancel{p(\mathbf{f}|\mathbf{u}, \mathbf{X}, \mathbf{Z})}p(\mathbf{u}|\mathbf{Z})}{\cancel{p(\mathbf{f}|\mathbf{u}, \mathbf{X}, \mathbf{Z})}q(\mathbf{u})} \\
&= \langle \log p(\mathbf{y}|\mathbf{f}) \rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X},\mathbf{Z})q(\mathbf{u})} - \mathsf{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u}|\mathbf{Z})\right) \\
&= \left\langle \log \mathcal{N}\left(\mathbf{y}|\mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{u}, \sigma^2\mathbf{I}\right) \right\rangle_{q(\mathbf{u})} - \mathsf{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u}|\mathbf{Z})\right)
\end{aligned}$$

Special Variational Posterior

- There is no inversion of any big covariance matrices in the first term:

$$-\frac{N}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\left\langle(\mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{u}-\mathbf{y})^\top(\mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{u}-\mathbf{y})\right\rangle_{q(\mathbf{u})}$$

- The overall complexity of the lower bound is $O(NM^2)$.

Tighten the Bound

- Find the optimal parameters of $q(\mathbf{u})$:

$$\mu^*, \Sigma^* = \arg\max_{\mu,\Sigma} \mathcal{L}(\mu, \Sigma).$$

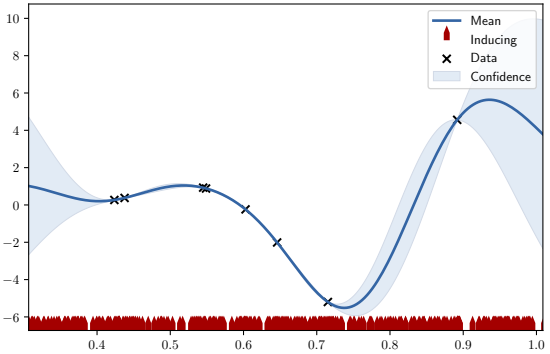
- Make the bound as tight as possible by plugging in μ^* and Σ^*:

$$\mathcal{L} = \log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^\top + \sigma^2\mathbf{I}\right) - \frac{1}{2\sigma^2}\text{tr}\left(\mathbf{K}_{ff} - \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^\top\right).$$

- The 1st term is the same as in the Nyström approximation.
- The overall complexity of the lower bound remains $O(NM^2)$.

Variational sparse GP

- Note that $\mathcal{L}$ is not a valid log-pdf, $\int_{\mathbf{y}} \exp(\mathcal{L}(\mathbf{y})) \leq 1$, due to the trace term.
- As inducing points are variational parameters, optimizing the inducing inputs $\mathbf{Z}$ always leads to a better bound.
- The model does not "overfit" with too many inducing points.

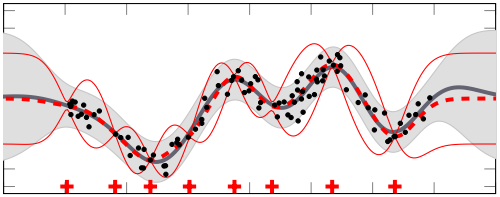


FITC vs. Variational sparse GP

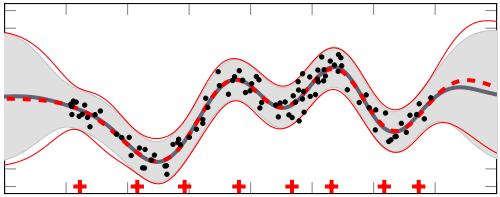
- model approximation vs. approximate inference (see [Bauer et al., 2016])
- Note that, when point estimating hyper-parameters, if the number of inducing points is too small, the model may "under-fit":

$$\mathcal{L} = \log p(y) - \mathrm{KL}\left(q(x) \,\|\, p(x|y)\right).$$

FITC (nlml $= 23.16$, $\sigma_n = 1.93 \cdot 10^{-4}$)



VFE (nlml $= 38.86$, $\sigma_n = 0.286$)



Optimal values for the exact GP: nlml $= 34.15$, $\sigma = 0.274$. [Bauer et al., 2016]

Limitations of Sparse GP

Variational sparse GP has computational complexity $O(NM^2)$.

The computation becomes infeasible under two scenarios:

- The number of data points N is very high, e.g., millions of data points.
- The function is very complex, which requires tens of thousands of inducing points.

Mini-batch Learning (1)

- Mini-batch learning allows DNNs to be trained on millions of data points.
- Given a set of inputs and labels, $\mathcal{D} = \{\mathbf{x}_i, y_i\}_{i=1}^N$, $(\mathbf{x}_i, y_i) \sim p(\mathbf{x}, y)$, the true loss function is defined as

$$c_{\text{true}} = \int l(f_\theta(\mathbf{x}), y) p(\mathbf{x}, y) \mathrm{d}\mathbf{x} \mathrm{d}y \approx \frac{1}{N} \sum_{i=1}^{N} l(f_\theta(\mathbf{x}), y) = c,$$

  where $f_\theta(\cdot)$ is DNN and $l(\cdot, \cdot)$ is the loss function.
- Gradient descent (GD) updates the parameters by

$$\theta_{t+1} = \theta_t - \eta \frac{\mathrm{d}c}{\mathrm{d}\theta}.$$

Mini-batch Learning (2)

- Mini-batch learning approximates the loss by subsampling the data,

$$c_{\mathsf{MB}} = \frac{1}{B} \sum_{\mathbf{x}_i, y_i \sim \tilde{p}(\mathbf{x}, y)} l(f_\theta(\mathbf{x}_i), y_i).$$

- Stochastic gradient descent (SGD) updates the parameters by

$$\theta_{t+1} = \theta_t - \eta \frac{\mathsf{d}c_{\mathsf{MB}}}{\mathsf{d}\theta}.$$

- Can mini-batch learning be applied to GPs as well?

Mini-batch Learning for GPs

- Mini-batch learning relies on the objective being an expectation w.r.t. the data, *i.e.*, $\langle l(f_\theta(\mathbf{x}), y) \rangle_{p(\mathbf{x},y)}$.
- The log-marginal likelihood of GP:

$$\log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K} + \sigma^2 \mathbf{I}\right)$$

- The variational lower bound of sparse GP:

$$\log \mathcal{N}\left(\mathbf{y}|0, \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^\top + \sigma^2 \mathbf{I}\right) - \frac{1}{2\sigma^2}\mathrm{tr}\left(\mathbf{K}_{ff} - \mathbf{K}_{fu}\mathbf{K}_{uu}^{-1}\mathbf{K}_{fu}^\top\right)$$

"Uncollapsed" Lower Bound

- Hensman et al. [2013] discovers that the "uncollapsed" variational lower bound of sparse GP can be used for mini-batch learning.
- The "uncollapsed" variational lower bound of sparse GP:

$$\mathcal{L} = \langle \log p(\mathbf{y}|\mathbf{f}) \rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X},\mathbf{Z})q(\mathbf{u})} - \mathrm{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u})\right)$$

- The 2nd term, $\mathrm{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u})\right)$, does not depend on the data.

"Uncollapsed" Lower Bound

- In the 1st term, as $p(\mathbf{y}|\mathbf{f}) = \mathcal{N}(\mathbf{y}|\mathbf{f}, \sigma^2\mathbf{I})$,

$$\log p(\mathbf{y}|\mathbf{f}) = \sum_{n=1}^{N} \log \mathcal{N}\left(y_n|f_n, \sigma^2\right)$$

- Denote $q(\mathbf{f}|\mathbf{X}, \mathbf{Z}) = \int p(\mathbf{f}|\mathbf{u}, \mathbf{X}, \mathbf{Z}) q(\mathbf{u}) \mathrm{d}\mathbf{u}$.

$$\begin{aligned}\langle \log p(\mathbf{y}|\mathbf{f}) \rangle_{q(\mathbf{f}|\mathbf{X},\mathbf{Z})} &= \left\langle \sum_{n=1}^{N} \log \mathcal{N}\left(y_n|f_n, \sigma^2\right) \right\rangle_{q(\mathbf{f}|\mathbf{X},\mathbf{Z})} \\ &= \sum_{n=1}^{N} \left\langle \log \mathcal{N}\left(y_n|f_n, \sigma^2\right) \right\rangle_{q(f_n|\mathbf{x}_n,\mathbf{Z})}\end{aligned}$$

Stochastic Variational GP (SVGP)

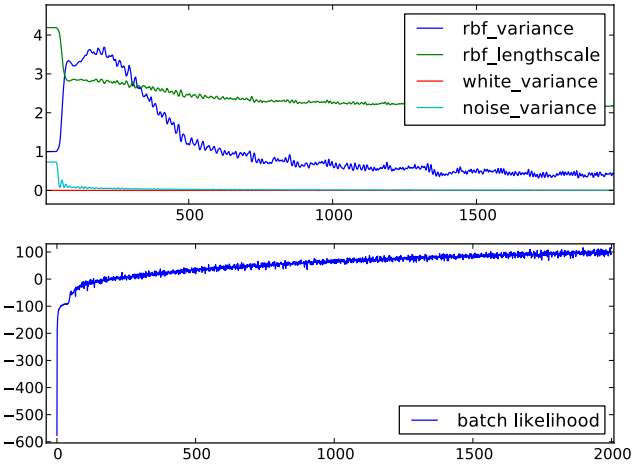
- The resulting lower bound can be written as the sum over the data,

$$\begin{aligned}\mathcal{L} =& \sum_{n=1}^{N} \left\langle \log \mathcal{N}\left(y_n | f_n, \sigma^2\right) \right\rangle_{q(f_n|\mathbf{x}_n,\mathbf{Z})} - \mathsf{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u})\right) \\ \approx& \frac{N}{B} \sum_{\mathbf{x}_i, y_i \sim \tilde{p}(\mathbf{x},y)} \left\langle \log \mathcal{N}\left(y_i | f_i, \sigma^2\right) \right\rangle_{q(f_i|\mathbf{x}_i,\mathbf{Z})} - \frac{N}{B}\mathsf{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u})\right) = \mathcal{L}_{\mathsf{MB}}\end{aligned}$$

- This allows us to do mini-batch learning with SGD,

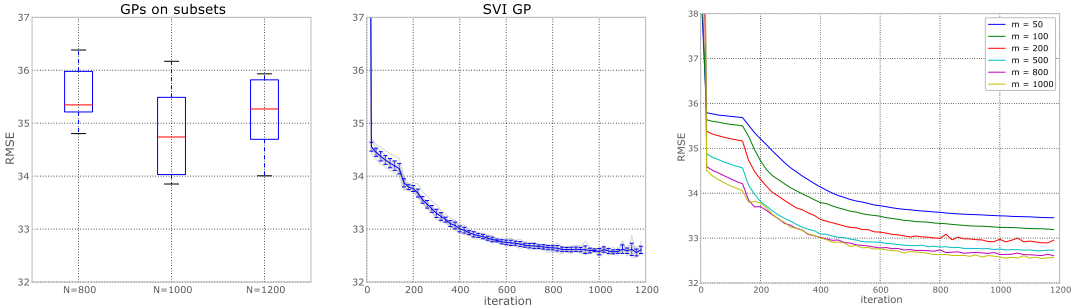
$$\theta_{t+1} = \theta_t - \eta \frac{\mathsf{d}\mathcal{L}_{\mathsf{MB}}}{\mathsf{d}\theta}.$$

2D Synthetic Data



Airline Delay Data

Flight delays for every commercial flight in the USA from January to April 2008.
700,000 train, 100,000 test



The pros and cons of SVGP

Pros

- With mini-batch learning, the computational complexity reduces from $O(NM^2)$ to $O(M^3)$.

Cons

- The variational distribution $q(\mathbf{u})$ needs to be explicitly optimized.
- The number of variational parameters increase from MQ to $(2M + M^2)Q$.
- Optimization relies on SGD methods and the methods like L-BFGS are no longer applicable.
- It can be challenging to initialize $q(\mathbf{u})$.

Non-Gaussian likelihood

So far, we have only discussed GP regression with Gaussian noise distribution.

In practice, many difference noise distributions for modelling real data, e.g.,

- Student-t distribution: data with outliers
- Poisson / Multi-nomial distribution: Integer counts
- Beta distribution: bounded real values
- Bernoulli / Categorical distribution: classification labels

Common approaches

Common choice for approximation inference:

- Exact GP with Laplace approximation
- Expectation Propagation (EP) with sparse GP

Both of them are quite complex to implement and difficult to scale.

SVGP with non-Gaussian likelihood

Let's use a binary classification as an example.

- The outputs are binary. $\mathbf{y} = (y_1, \ldots, y_N)$, $y_i \in \{0, 1\}$.
- The likelihood is a Bernoulli distribution with a Sigmoid link function:

$$p(y_i|f_i) = \sigma(f_i)^{y_i}(1 - \sigma(f_i))^{(1-y_i)}$$

SVGP with non-Gaussian likelihood

- The lower bound of SVGP is

$$\mathcal{L} = \sum_{n=1}^{N} \left\langle \log p(y_n|f_n) \right\rangle_{q(f_n|\mathbf{x}_n,\mathbf{Z})} - \mathsf{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u})\right).$$

- The 2nd term, $\mathsf{KL}\left(q(\mathbf{u}) \,\|\, p(\mathbf{u})\right)$, is closed form.
- The 1st term, $\sum_{n=1}^{N} \left\langle \log p(y_n|f_n) \right\rangle_{q(f_n|\mathbf{x}_n,\mathbf{Z})}$, is the sum of a list of 1D integrals.
- Those integrals are intractable.

Gauss-Hermite Quadrature

Regarding those 1D integrals,

- $q(f_n|\mathbf{x}_n, \mathbf{Z})$ is a 1D Gaussian distribution. See the definition:

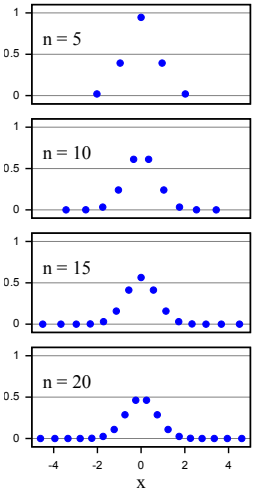
$$q(f_n|\mathbf{x}_n, \mathbf{Z}) = \int p(f_n|\mathbf{u}, \mathbf{x}_n, \mathbf{Z}) q(\mathbf{u}) \mathrm{d}\mathbf{u}.$$

- Gauss-Hermite quadrature can be applied,

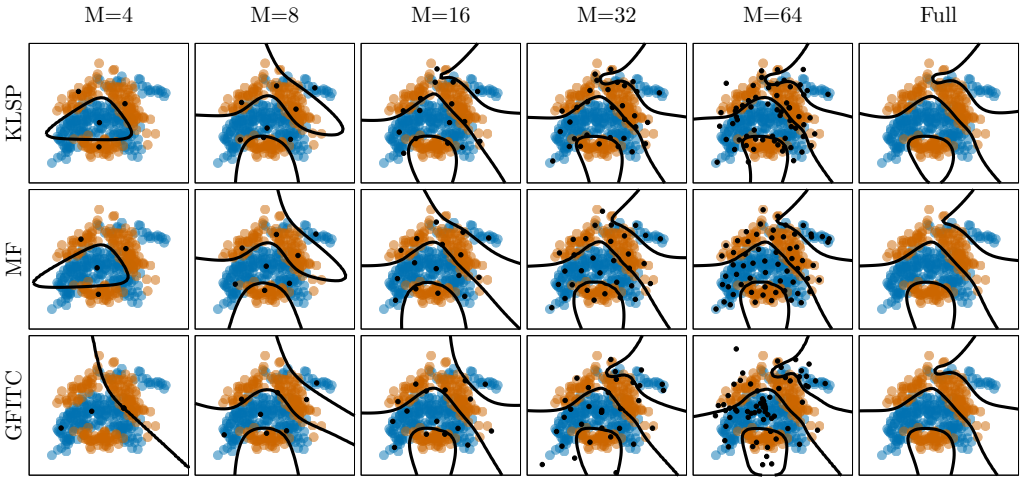
$$\langle \log p(y_n|f_n) \rangle_{q(f_n|\mathbf{x}_n, \mathbf{Z})} \approx \sum_{j=1}^{C} w_j \log p(y_n|f_j),$$

$$w_j = \frac{2^{C-1} C! \sqrt{\pi}}{C^2 [H_{C-1}(f_j)]^2}.$$

- The quadrature result is exact if $\log p(y_n|f_n)$ is a polynomial with its order less than C.



Binary Classification Example



[Hensman et al., 2015]

Beyond 1D GPs

- Multi-class classification is a common example.
- For a C-class classification, $y \in \{1, \ldots, C\}$, a GP is used to model each class,

$$\mathbf{f}_1, \ldots, \mathbf{f}_C \sim GP(0, K(\mathbf{X}, \mathbf{X})).$$

- The common likelihood is a categorical distribution with a soft-max function,

$$p(y_n | f_{n1}, \ldots, f_{nC}) = \prod_{j=1}^{C} g(f_{nj})^{\delta[y_n - j]}, \; g(f_{nj}) = \frac{e^{f_{nj}}}{\sum_{j'=1}^{C} e^{f_{nj'}}}$$

- Gauss-Hermite quadrature is not a good choice due to high dimensionality.

Monte Carlo Sampling

- Monte Carlo sampling can approximate the multi-dimensional integral:

$$\begin{aligned}\langle \log p(y_n|\mathbf{f}_n)\rangle_{q(\mathbf{f}_n|\mathbf{x}_n,\mathbf{Z})} &= \int q(\mathbf{f}_n|\mathbf{x}_n,\mathbf{Z}) \sum_{j=1}^{C} \delta[y_n - j] \log g(f_{nj}) \mathrm{d}\mathbf{f}_n \\ &\approx \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{C} \delta[y_n - j] \log g(f_{tnj})\end{aligned}$$

  where $\mathbf{f}_{tn} \sim q(\mathbf{f}_n|\mathbf{x}_n,\mathbf{Z})$ and $\mathbf{f}_{tn} = (f_{tn1},\ldots,f_{tnC})$.

- *Reparameterization trick* can be used to reduce of the variance of the gradient. Denote $q(f_{nj}|\mathbf{x}_{nj},\mathbf{Z}) = \mathcal{N}(m_{nj},\sigma_{nj}^2)$. A sample can be rewritten as

$$f_{tnj} = m_{nj} + \sigma_{nj}\epsilon_t,\ \epsilon_t \sim \mathcal{N}(0,1).$$

Alternative Approach

- Although *reparameterization trick* makes Monte Carlo sampling usable for the above case, the gradient variance can be quite high if C is big or the uncertainty is high.
- This leads to slow convergence.
- proposes a smart formulation for a special case of multi-class classification.

Reformulation of Multi-class Classification (1)

- $p(y|\mathbf{f})$ returns 1 if g_y is the biggest one in $\mathbf{f}$, otherwise 0,

$$p(y|\mathbf{f}) = \prod_{j\in\{1,\dots,C\}, j\neq y} [(1-\epsilon)^{H(f_j-f_y)}\epsilon^{1-H(f_j-f_y)}],\ H(f) = \begin{cases} 1, & f \leq 0, \\ 0, & f > 0. \end{cases}$$

- The 1st term $\langle \log p(y_n|\mathbf{f}_n) \rangle_{q(\mathbf{f}_n|\mathbf{x}_n,\mathbf{Z})}$ becomes

$$\left\langle \sum_{j\in\{1,\dots,C\}, j\neq y} [H(f_j - f_y)\log(1-\epsilon) + (1 - H(f_j - f_y))\log\epsilon] \right\rangle_{q(\mathbf{f}_n|\mathbf{x}_n,\mathbf{Z})}$$
$$= \sum_{j\in\{1,\dots,C\}, j\neq y} \log\frac{1-\epsilon}{\epsilon} \langle H(f_j - f_y)\rangle_{q(f_y,f_j|\mathbf{x}_n,\mathbf{Z})} + \log\epsilon$$

Reformulation of Multi-class Classification (2)

- Denote $q(f_i|\mathbf{x}_n, \mathbf{Z}) = \mathcal{N}(m_i, \sigma_i)$.
- The 1st term becomes

$$\langle \log p(y_n|\mathbf{f}_n) \rangle_{q(\mathbf{f}_n|\mathbf{x}_n,\mathbf{Z})} = \left\langle \sum_{j\in\{1,\dots,C\}, j\neq y} \log \frac{1-\epsilon}{\epsilon} \Phi(\frac{f_y - m_j}{\sigma_j}) \right\rangle_{q(f_y|\mathbf{x}_n,\mathbf{Z})} + \log \epsilon$$

  where $\Phi(\cdot)$ is the CDF of Gaussian distribution.
- It is an 1D integral now and the Gaussi-Hermite quadrature is applicable again!

Are big covariance matrices always (almost) low-rank?

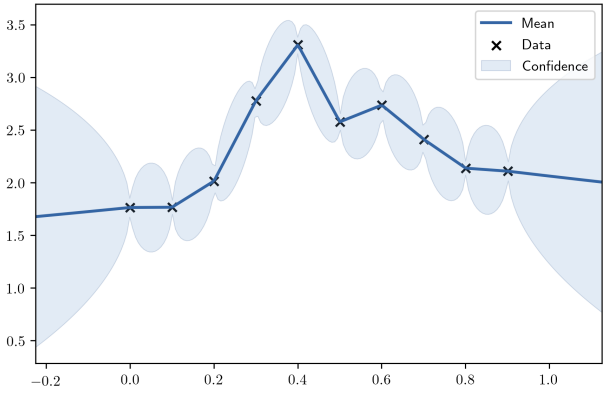
- Of course, not.
- A time series example

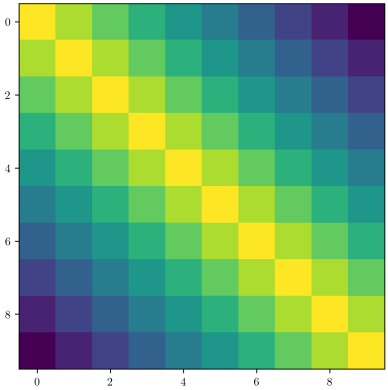
$$y = f(t) + \epsilon$$

- The data are collected with even time interval continuously.

A time series example: 10 data points

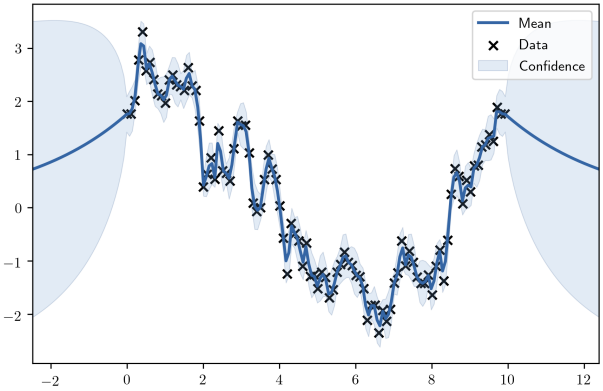
When we observe until $t = 1.0$:

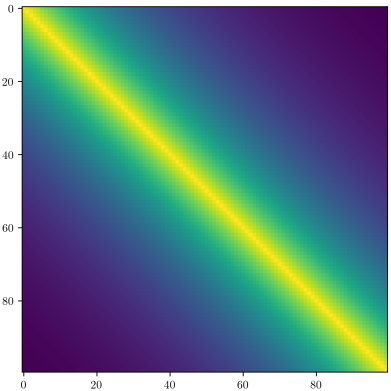




A time series example: 100 data points

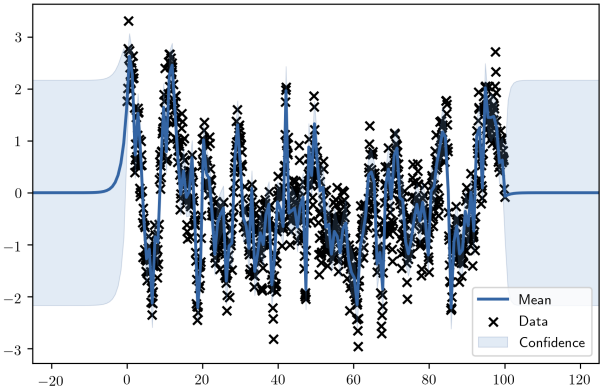
When we observe until $t = 10.0$:

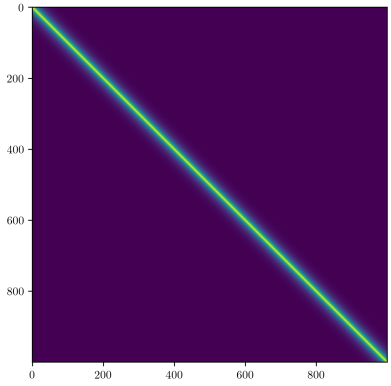




A time series example: 1000 data points

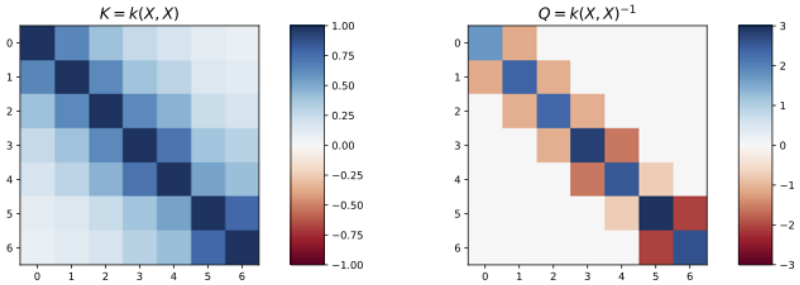
When we observe until $t = 100.0$:





Banded precision matrix

- For the kernels like the Matern family, the precision matrix is banded.
- For example, given a Matern $\frac{1}{2}$ or known as exponential kernel: $k(x, x') = \sigma^2 \exp(-\frac{|x-x'|}{l^2})$.



This slide is taken from Nicolas Durrande [Durrande et al., 2019].

Closed form precision matrix

- The precision matrix of Matern kernels can be computed in closed form.
- The lower triangular matrix from the Cholesky decomposition of the precision matrix is banded as well.

$$\log(\mathbf{y}|\mathbf{X}) = -\frac{1}{2}\log|2\pi(LL^\top)^{-1}| - \frac{1}{2}\mathsf{tr}\left(\mathbf{y}\mathbf{y}^\top LL^\top\right)$$

  where L is the lower triangular matrix from the Cholesky decomposition of the precision matrix Q, $Q = LL^\top$.
- The computational complexity becomes $O(N)$.

Other approximations

- deterministic/stochastic frequency approximation
- distributed approximation
- conjugate gradient methods for covariance matrix inversion

Q & A!

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